FAKULTÄT FÜR !NFORMATIK
Faculty of Informatics

Diplomarbeitspräsentation

Cyclic Superposition and Induction

Masterstudium:

Logic and Computation

Jannik Vierling

Technische Universität Wien
Institut für Diskrete Mathematik und Geometrie
Arbeitsbereich: Computational Logic
Betreuer: Associate Prof. Dipl.-Ing. Dr.techn.Stefan Hetzl

Background and Motivation

Inductive theorem proving is a branch of automated deduction that aims at mechanizing the process of proving mathematical statements involving some notion of induction.

The most notable applications of inductive theorem proving are the formal verification of hardware and software as well as the formalization of mathematical proofs.

The n-clause calculus is an approach to inductive theorem proving that was introduced in 2013 by Kersani and Peltier [KP13]. This approach consists in enhancing a superposition calculus by a cycle detection mechanism. The cycles detected by this calculus represent a form of mathematical induction. Presently it is not well understood, which sentences are provable with this method.

Superposition Calculus

The superposition calculus is a *refutational* calculus for equational logics. Its central inference rule is the superposition inference.

$$\frac{C \vee t \bowtie s \quad D \vee u \simeq v}{(C \vee D \vee t[v]_p \bowtie s)\sigma}\ (\text{sup})\ \sigma = \mathbf{mgu}(t|_p, u)$$

This calculus is usually implemented as a *saturation* procedure which exhaustively applies the superposition inference above until the empty clause is eventually derived.

Inductive Cycles

The n-clause calculus detects *inductive cycles* during the saturation process. An inductive cycle is a set of clauses of clauses $S(n)$ that satisfies the conditions below, where n is a parameter representing a natural number.

$$S(n) \models n \neq 0 \quad (1)$$

$$S(n) \models S(n-1) \quad (2)$$

Induction cycles represent arguments by infinite descent, where condition (1) corresponds to the base case, and (2) corresponds to the step case (see Figure 2). Therefore, an inductive cycle $S(n)$ is an unsatisfiable clause set.

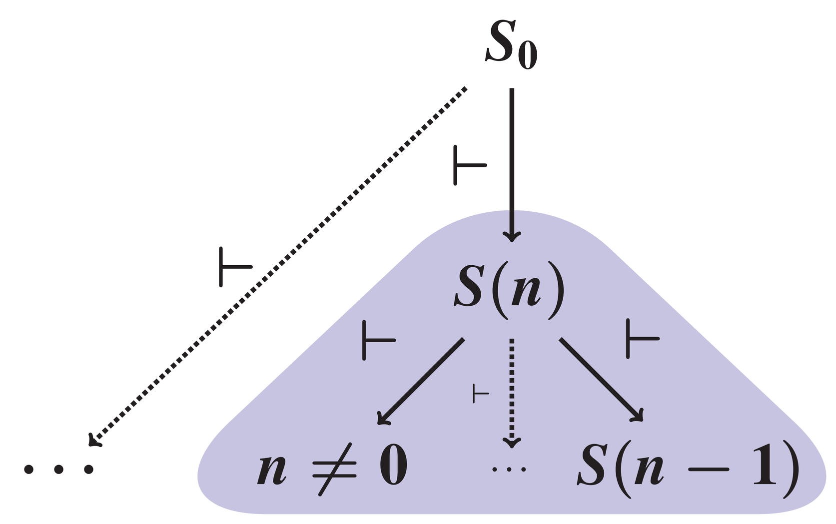

Figure 1: A cyclic superposition refutation of a clause set S_0 with inductive cycle $S(n)$.

Accordingly, a clause set S_0 is said to be refuted in the cyclic superposition calculus if it is possible to derive an inductive cycle $S(n)$ from it.

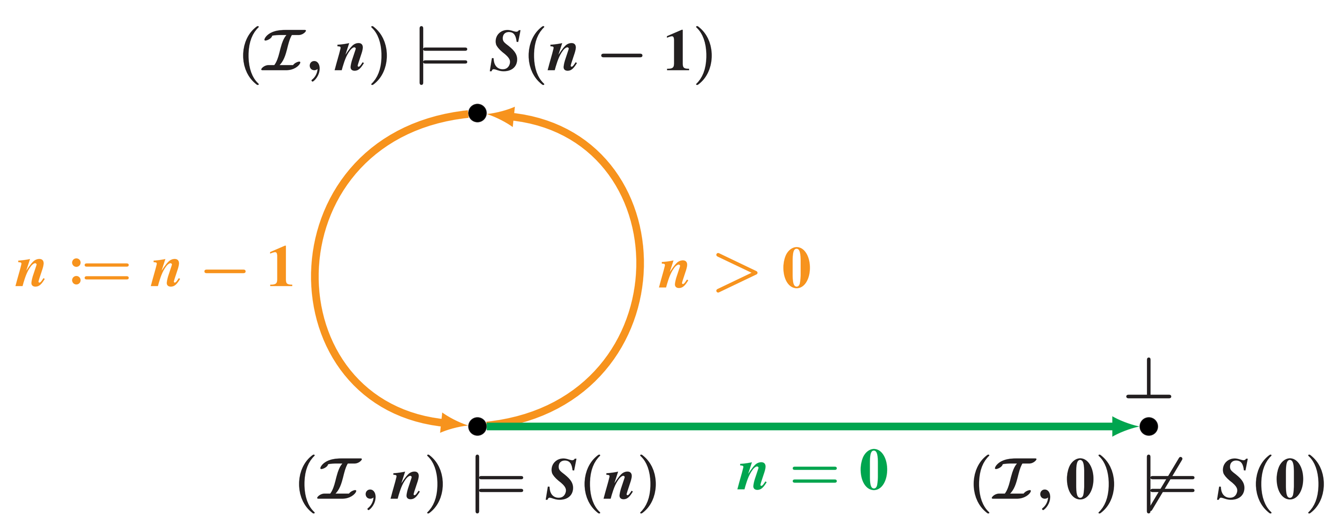

Figure 2: Argument by infinite descent represented by an inductive cycle $S(n)$, where $\mathcal{I}$ is a model of $S(n)$.

Objective of this Thesis

The objective of this thesis is to give an upper bound on the quantifier complexity of the induction invariants required to simulate the arguments realized by refutations of the n-clause calculus.

Approach

The problem is approached by a translation of n-clause refutations into the calculus **LK** with structural induction. The translation consists of two major steps. In the first step the cycles detected by the n-clause calculus are translated into the cyclic sequent calculus **CLKID**$^{\omega}$ introduced by Brotherston and Simpson in [BS10] (see Figure 3).

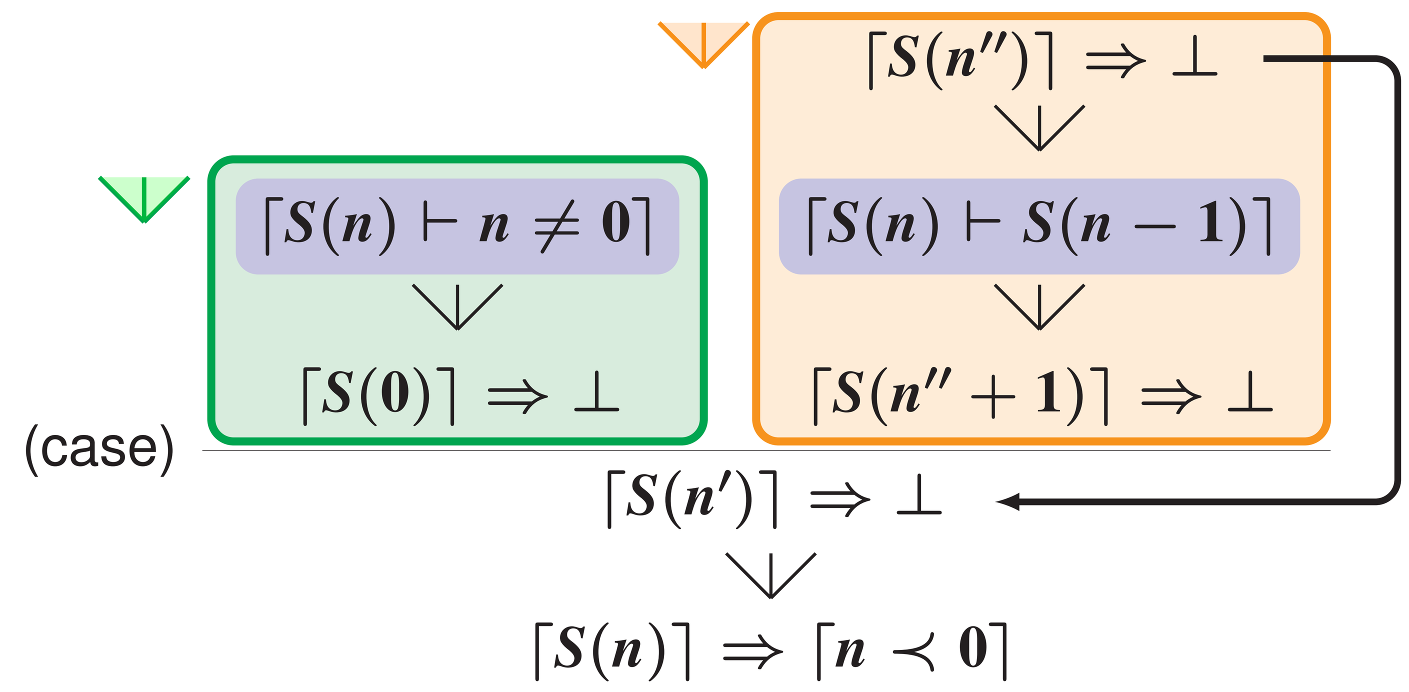

Figure 3: Cyclic proof representation of the inductive cycle shown in Figure 1.

In a second step the cyclic proofs obtained in the first step are translated into the calculus **LK** with structural induction (see Figure 4). Finally, suitable axioms are added in order to establish the unsatisfiability of the statement.

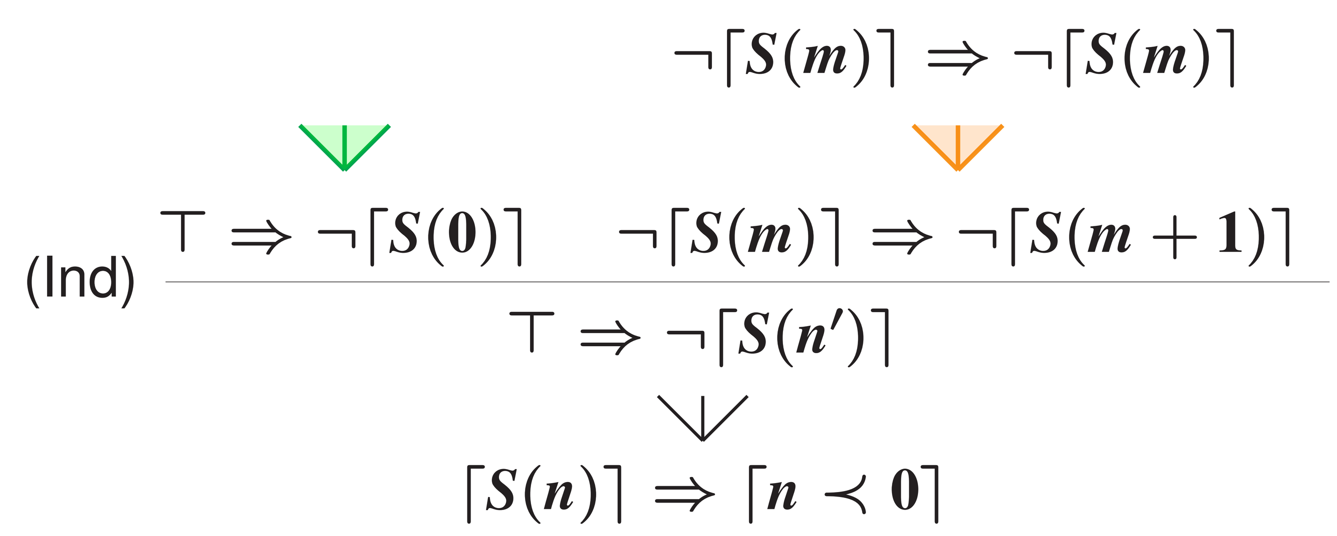

Figure 4: Inductive proof corresponding to the cyclic proof shown in Figure 3.

Main Result

By the translation outlined above we obtain an upper bound on the quantifier complexity of the induction invariants captured by the notion of inductive cycle.

Theorem *If a clause set* $S(n)$ *admits a cyclic superposition refutation, then the sequent* $\lceil S(n) \rceil \Rightarrow \bot$ *is provable in* **LK** *with* Σ_1*-induction.*

References

- James Brotherston and Alex Simpson.
  Sequent calculi for induction and infinite descent.
  Journal of Logic and Computation, 21(6):1177–1216, 2010.
- Abdelkader Kersani and Nicolas Peltier.
  Combining superposition and induction: A practical realization.
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Kontakt: jannik.vierling@tuwien.ac.at